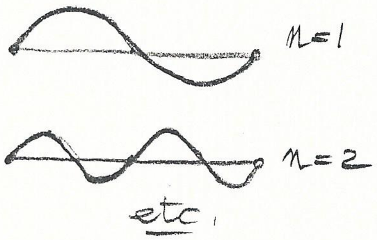
140A Problem Set 4 Suggested Answers

[1] Question 1

3.11 $\Rightarrow$ $q = n \cdot \frac{2\pi}{L}$, $q = \frac{2\pi}{\lambda}$

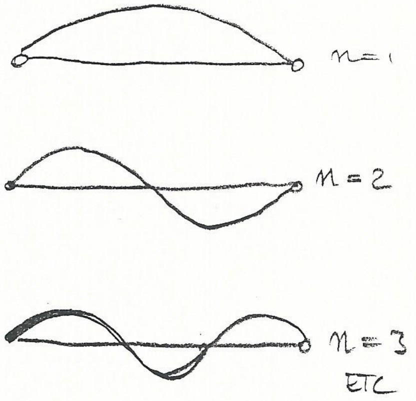
$$\Rightarrow \quad n \cdot \frac{\cancel{2\pi}}{L} = \frac{\cancel{2\pi}}{\lambda} \Rightarrow \boxed{L = n\lambda}$$



for A VIBRATING STRING, THINGS ARE DIFFERENT. IN FACT, if YOU TAKE A STRING AND CONSTRAIN THE TWO END POINTS, YOU EXPECT A SOLUTION of THE form A sin qx. NOW, if YOU fix THE END POINT TO BE ZERO,

$$\Rightarrow \sin qx\big|_{x=L} = 0 \Rightarrow \sin qL = 0 \Rightarrow qL = n\pi$$

SO THAT $\frac{\cancel{2\pi} L}{\lambda} = n\cancel{\pi} \Rightarrow \boxed{L = n\frac{\lambda}{2}}$ WHICH SHOULD BE A FAMILIAR RESULT from GENERAL PHYSICS



THE REASON WHY THEY ARE DIFFERENT, DESPITE THE fact THAT THE SAME WAVE EQN' IS INVOLVED, IS BECAUSE WE IMPOSED different BOUNDARY CONDITIONS!

VIBRATING STRING = EXTREMA @ ZERO, DON'T MOVE

DERIVATION of 3.11 = PERIODIC BOUNDARY CONDITIONS

[2] Question 4

YOU DO NOT EXPECT SOUND WAVES IN MOLECULES
YOU WOULD NOT BE ABLE TO DEFINE A STRESS
SINCE THE MOLECULES ARE NOT LINKED TOGETHER.

OF COURSE, YOU CAN HAVE PROPAGATION OF SOUND IN GASES, BUT THESE ARE PRESSURE WAVES, WAVES WHICH PROPAGATE A LOCAL DISTURBANCE OF THE GAS DENSITY. SOUND WAVES ARE NOT TRANSVERSAL, THEY ARE LONGITUDINAL.

IN A SOLID YOU CAN HAVE BOTH LONGITUDINAL (COMPRESSIONAL) AND TRANSVERSE (SHEAR) WAVES

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[3] 3.2

both longitudinal (compressional) and transverse (shear) waves can be transmitted by the solid, and since their speeds are different, we replace v_s

IN (3.20) $g(\omega) = \frac{3V}{2\pi^2}\frac{\omega^2}{v_s^3}$ by a more general ex-

pression. In particular, if v_l represents the speed of longitudinal waves and v_t the speed of transverse waves in the solid, we require $3N_0 = (4\pi V/3v_l^3)v_m^3 + (4\pi V/3v_t^3)2v_m^3$ instead of (11-27a), where allowance is now made for the two polarization states of transverse waves, as well. Then we use in [redacted] (3.20)

$$\frac{1}{v^3} = \left(\frac{1}{v_l^3} + \frac{2}{v_t^3}\right)$$

Note the extra factor of 2 for the transverse waves in the equation summing modes to yield 3No

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$$\bar{\mathcal{E}} = \frac{\int_0^\infty \mathcal{E} e^{-\mathcal{E}/kT} d\mathcal{E}}{\int_0^\infty e^{-\mathcal{E}/kT} d\mathcal{E}}$$

3.4
Not assigned this year

$$\int_0^\infty e^{-\mathcal{E}/kT} d\mathcal{E} = -kTe^{-\mathcal{E}/kT}\Big|_0^\infty = kT \quad \checkmark$$

$$\int_0^\infty \mathcal{E} e^{-\mathcal{E}/kT} d\mathcal{E} = \text{let } \frac{\mathcal{E}}{kT} = x, \quad d\mathcal{E} = kT\,dx \quad \Rightarrow$$

$$\int_0^\infty \mathcal{E} e^{-\frac{\mathcal{E}}{kT}} d\mathcal{E} = (kT)^2 \int_0^\infty x e^{-x} dx = (kT)^2 \left[-(1+x)e^{-x} \Big|_0^\infty \right] = (kT)^2$$

$$\therefore \quad \bar{\mathcal{E}} = \frac{(kT)^2}{kT} = kT \quad \checkmark$$

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[4] 3.5

Prove that $\bar{\mathcal{E}} = \dfrac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}$

OMAR P.77 SUGGESTS HOW TO DERIVE THIS RESULT.

YOU MAY ALSO CONSIDER THIS MORE CONVENTIONAL APPROACH

▶The quantity $\bar{\mathcal{E}}$ is evaluated from the ratio of sums

$$\bar{\mathcal{E}} = \frac{\sum_{n=0}^{\infty} \mathcal{E} P(\mathcal{E})}{\sum_{n=0}^{\infty} P(\mathcal{E})}$$

analogous to the ratio of integrals in (1-21). Sums must be used because with Planck's postulate the energy $\mathcal{E}$ becomes a discrete variable that takes on only the values $\mathcal{E} = 0, h\nu, 2h\nu, 3h\nu, \ldots$. That is, $\mathcal{E} = nh\nu$ where $n = 0, 1, 2, 3, \ldots$. Evaluating the Boltzmann distribution $P(\mathcal{E}) = e^{-\mathcal{E}/kT}/kT$, we have

$$\bar{\mathcal{E}} = \frac{\sum_{n=0}^{\infty} \frac{nh\nu}{kT} e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} \frac{1}{kT} e^{-nh\nu/kT}} = kT \frac{\sum_{n=0}^{\infty} n\alpha e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}} \qquad \text{where } \alpha = \frac{h\nu}{kT}$$

This, in turn, can be evaluated most easily by noting that

$$-\alpha \frac{d}{d\alpha} \ln \sum_{n=0}^{\infty} e^{-n\alpha} = \frac{-\alpha \frac{d}{d\alpha} \sum_{n=0}^{\infty} e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}} = \frac{-\sum_{n=0}^{\infty} \alpha \frac{d}{d\alpha} e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}} = \frac{\sum_{n=0}^{\infty} n\alpha e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}}$$

so that

$$\bar{\mathcal{E}} = kT\left(-\alpha \frac{d}{d\alpha} \ln \sum_{n=0}^{\infty} e^{-n\alpha}\right) = -h\nu \frac{d}{d\alpha} \ln \sum_{n=0}^{\infty} e^{-n\alpha}$$

Now

$$\sum_{n=0}^{\infty} e^{-n\alpha} = 1 + e^{-\alpha} + e^{-2\alpha} + e^{-3\alpha} + \cdots$$

$$= 1 + X + X^2 + X^3 + \cdots \qquad \text{where } X = e^{-\alpha}$$

but

$$(1 - X)^{-1} = 1 + X + X^2 + X^3 + \cdots$$

so

$$\bar{\mathcal{E}} = -h\nu \frac{d}{d\alpha} \ln (1 - e^{-\alpha})^{-1}$$

$$= \frac{-h\nu}{(1 - e^{-\alpha})^{-1}} (-1)(1 - e^{-\alpha})^{-2} e^{-\alpha}$$

$$= \frac{h\nu e^{-\alpha}}{1 - e^{-\alpha}} = \frac{h\nu}{e^{\alpha} - 1} = \frac{h\nu}{e^{h\nu/kT} - 1}$$

$h\nu = \hbar\omega$!

✓

[5] 3.6

$$\Theta_D^{Cu} = 340\ K$$

a) THE CLASSICAL EXPRESSION for THE ENERGY PER MOLE IS GIVEN BY (3,23) IN OMAR, NAMELY

$$E = 3 N_A k T = 3RT$$

WE NEED $E(T)\Big|_{T=\Theta_D^{Cu}} =$

$$= 3 \cdot 6.023 \times \cancel{10^{23}} \times 1.381 \times \cancel{10^{-23}} \ \frac{Joule}{K} \times 340\ K \cong 8484.1\ Joules$$

b)

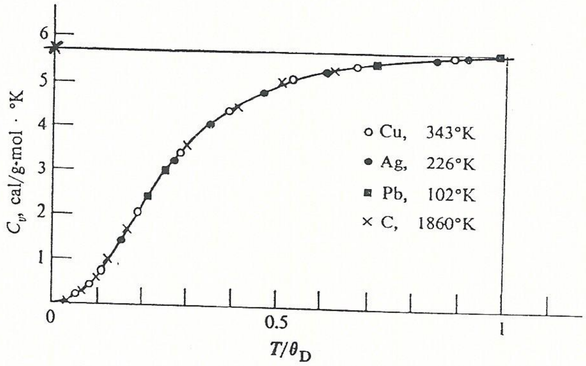


Fig. 3.13 Specific heats versus reduced temperature for four substances. Numbers refer to Debye temperatures. Note the high Debye temperature for diamond.

from THE fig. ABOVE, $C_v \simeq 5.7\ \frac{cal}{mol}\ \frac{1}{K}$

$$\therefore\ E_{MOLE} = 5.7\ \left(\frac{cal}{\cancel{K}}\right) \times 340\ \cancel{K} \cong 8112.4\ Joule,$$

close TO WHAT CALCULATED BEFORE

NOT A SURPRISE, from THE figure YOU CAN SEE THAT if $T \simeq \Theta_D \Rightarrow$ specific HEAT IS PRATICALLY GIVEN BY THE CLASSICAL EXPRESSION

4

c/

PER ATOM, $E = 3K_BT$ WHEN $T \sim \theta_D$

YOU CAN THINK THAT THIS ENERGY IS THEN TRANSFERED TO THE ATOM, WHICH IN TURN WILL OSCILLATE AROUND ITS EQUILIBRIUM POS'N

if THE OSCILLATIONS ARE HARMONIC $\Rightarrow$ HARMONIC OSCILLATOR

$$H = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_{MAX}^2$$

i.e, MAXIMUM DISPLACEMENT from EQUILIBRIUM IS WHEN ALL THE ENERGY IS STORED AS POTENTIAL.

$\frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2x^2$; TO ESTIMATE ω_0,

LET'S CONSIDER A SINGLE MODE of frequency ω_D, SO THAT $\hbar\omega_D = K_B\theta_D \Rightarrow \omega_D \sim \frac{K_B\theta_D}{\hbar}$

$\therefore$ TO ESTIMATE $\bar{x} = x_{MAX}$, JUST SET $\frac{1}{2}k\bar{x}^2$ EQUAL TO $3K_BT$ AND SOLVE for $\bar{x}$ $\Rightarrow$

$$\frac{1}{2}k\bar{x}^2 = \frac{1}{2}m\omega_D^2\bar{x}^2 = 3K_BT/_{\theta_D} \; ; \quad \bar{x}^2 = \frac{6K_B\theta_D}{m\omega_D^2} \cong \frac{6k_B\theta_D\hbar^2}{m\cdot(k_B\theta_D)^2}$$

$$\bar{x} \approx \sqrt{\frac{6\hbar^2}{mK_B\theta_D}}, \quad (\text{for } T = \theta_D) \; ;$$

TAKING for $m \cong 1{,}05\times10^{-25}$ kg (Cu-ATOM MASS) $\Rightarrow$

$$\bar{x} \approx \left(\frac{6\times(1.0545\times10^{-34})^2\left(\frac{Joule}{Jsec}\right)}{1.05\times10^{-25}\,kg\times1{,}381\times10^{-23}\,\frac{Joule}{K}}\;\frac{1}{340\,K}\right)^{1/2}$$

IN METERS! (MKS!)

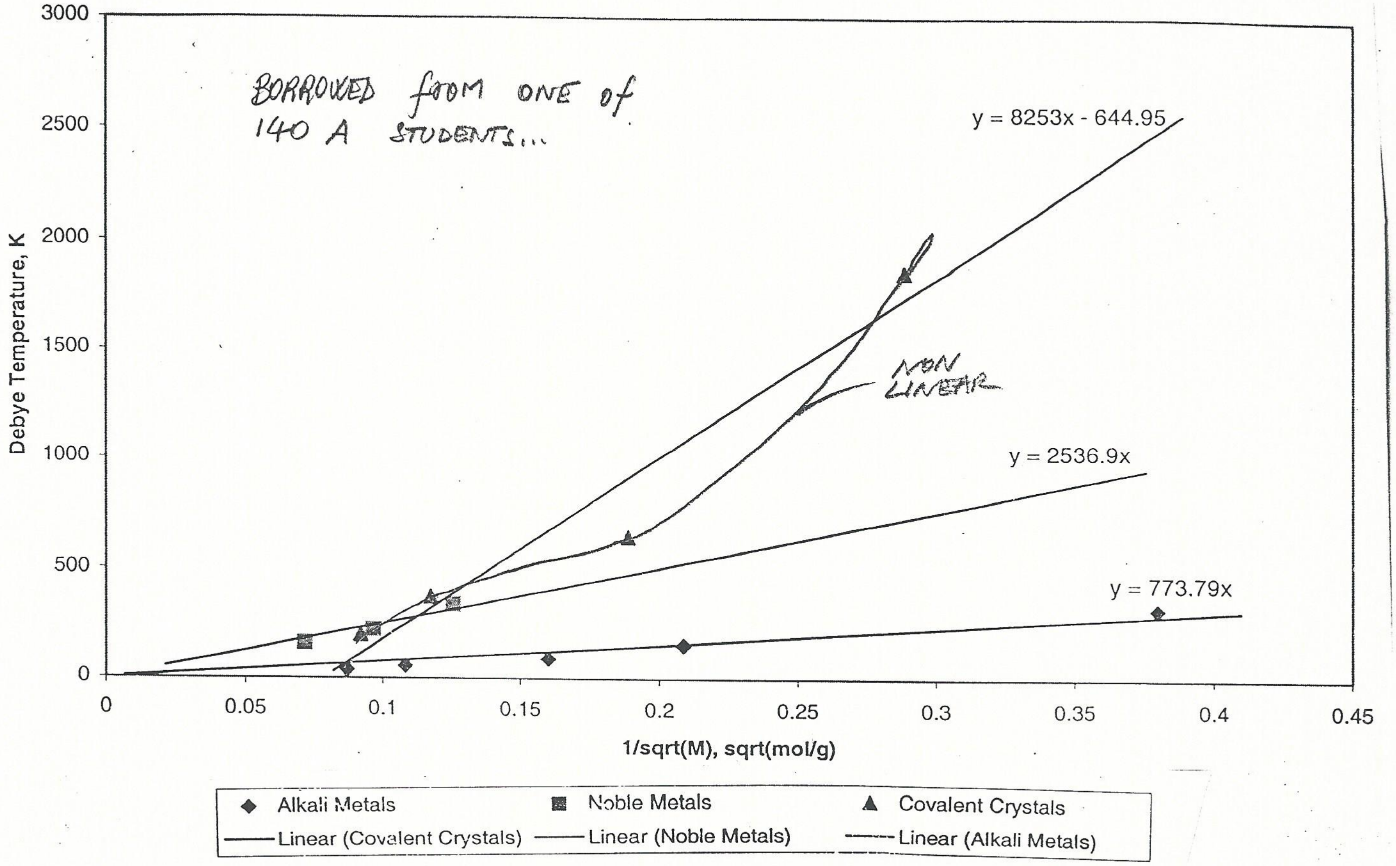
$\bar{x} \simeq 1{,}16\times10^{-11}\,m \simeq 0{,}11\,\text{Å}$

since for Cu the Lattice Constant is $\sim 3{,}61$ Å,

5 $\frac{\bar{x}}{a} \cong 0.032$! $\bar{x} \sim$ 3 % of the lattice constant

NB I TOOK $\bar{x} = r = \sqrt{x^2+y^2+z^2}$ $\Rightarrow$ $\langle x\rangle = \langle y\rangle = \langle z\rangle \simeq$ 1% of " " ✓

3.7 not assigned this year



BORROWED from ONE of 140 A STUDENTS...

if YOU ARE ASKED TO PLOT θ_D vs $M^{-1/2}$ AND CHECK WHETHER $\theta_D \propto M^{-1/2}$ => LOOK for STRAIGHT LINES!

ALKALI → ALL RIGHT, LINEAR
NOBLE METALS → OK, LINEAR
COVALENT CRYSTALS → NOT OK, NON LINEAR

from OMAR

Carbon & Lead are covalents crystals

$$\theta_D \sim \sqrt{\frac{Y}{\rho}}\, n^{1/3} \sim \sqrt{\frac{Y}{M}}, \qquad (3.39)$$

where Y is Young's modulus and M the atomic mass (using $\rho = nM$). Therefore θ_D depends primarily on the elastic constant of the substance Y and on the atomic mass M. The stiffer the crystal and the smaller M, the higher is θ_D. This explains, for instance, why θ_D is high for carbon (1860°K), which is stiff and light, and low for lead (102°K), which is soft and heavy.

[6] 3.10

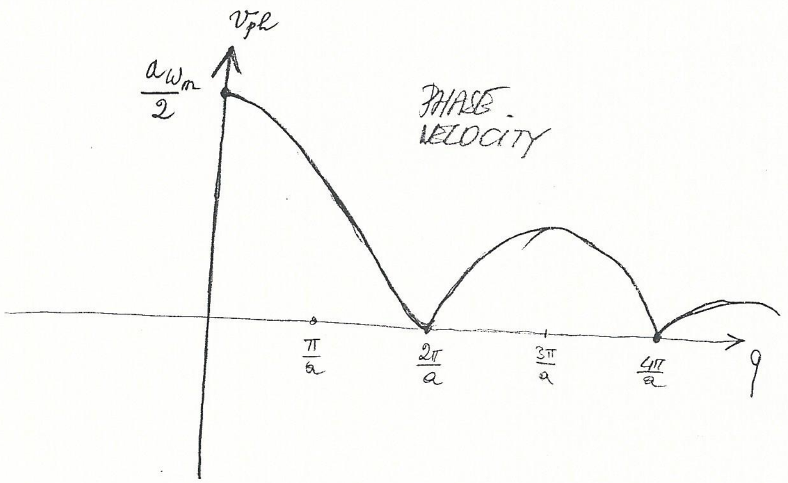
(3.46) IN OMAR

$$\omega = \omega_m \left| \sin(qa/2) \right|$$

DISPERSION RELATION FOR 1D ATOMIC CHAIN

phase velocity : $v_{ph} = \frac{\omega}{q} = \frac{\omega_m |\sin(qa/2)|}{q}$ NOT PERIODIC IN q

group velocity : $v_g = \frac{d\omega}{dq}$



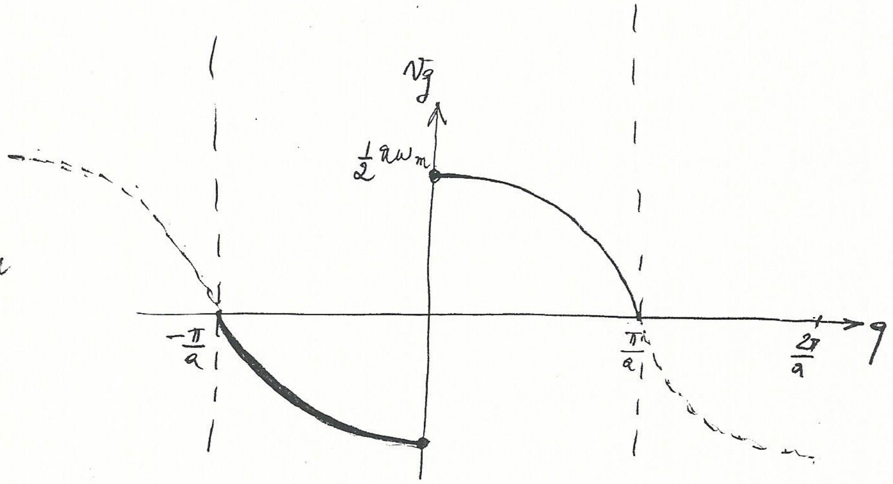
$$v_{ph} = \frac{\omega_m |\sin(qa/2)|}{q}$$

odd function, $v_{ph}(q) = -v_{ph}(-q)$
symmetric WRT origin

GROUP VELOCITY

$$\frac{d\omega}{dq} = \pm \frac{a\omega_m}{2} \cos(qa/2)$$

slope of the ω vs q dispersion curve in Fig 3.17



NEGATIVE VALUES FOR q DENOTE PROPAGATION IN THE OPPOSITE DIRECTION!

OMAR P. 93 gives a physical description about this problem.

NOTE THAT @ $q = 0$, $v_{ph} = v_g = \frac{1}{2} a\omega_m$

3.16 Not assigned this year

USE (3.74) $\Rightarrow$ $K = \frac{1}{3} C_v v \ell$ ← MEAN free PATH

C_v: SPECIFIC HEAT PER UNIT VOLUME

v: SOUND VELOCITY

for NaCl $k = 6{,}4 \ \frac{\text{Watt}}{\text{m K}}$, $67\ \text{Å}$ @ 213 K

$K = 45$ " , $2{,}3 \times 10^{-4}$ cm @ 20 K

DENSITY $= 2{,}18\ \text{g/cm}^3$

$\theta_D = 280$ K

first, compute $\frac{T}{\theta_D}$ SO THAT YOU KNOW HOW TO CALCULATE THE SPECIFIC HEAT.

$$\therefore \begin{cases} \frac{20}{\theta_D} \sim \frac{20}{280} \simeq 0{,}07 \ll 1 \Rightarrow \text{USE } \underline{\underline{3{,}40}} \\ \frac{273}{\theta_D} \sim 1 \Rightarrow \text{USE CLASSICAL DOULONG – PETIT RESULT} \end{cases}$$

LET'S CALCULATE C_{molar} first.

$$C_{@20} = \frac{12\pi^4}{5} R \left(\frac{T}{\theta_D}\right)^3 = \frac{12\pi^4}{5} \cdot \left(8.31 \ \frac{\text{Joule}}{\text{mol} \cdot \text{K}}\right) \simeq 0{,}707 \ \frac{\text{Joule}}{\text{mol} \cdot \text{K}}$$

$$C_{@273} = 3R = 3 \times 8{,}31 \ \frac{\text{Joule}}{\text{mol} \cdot \text{K}} = 24{,}93 \ \frac{\text{Joule}}{\text{mol} \cdot \text{K}}$$

1 mole of NaCl is EQUAL TO $(23 + 35.4)\,g \simeq 58.4\,g$ $\Rightarrow$

$\rho = 2{,}18\ \text{g/cm}^3 \Rightarrow$ VOLUME $= \frac{58.4}{2{,}18}\ \text{cm}^3 \cong 26.79\ \text{cm}^3$

$$C_v^{@20} = \frac{C_{@20}}{\text{VOLUME}} \simeq 0.0264 \ \frac{\text{Joule}}{\text{K cm}^3} = 2.64 \times 10^4 \ \frac{\text{Joule}}{\text{K m}^3}$$

$$C_v^{@273} = \frac{C_{@273}}{\text{VOLUME}} \simeq 0.93 \ \frac{\text{Joule}}{\text{K cm}^3} = 93 \times 10^4 \ \frac{\text{Joule}}{\text{K m}^3}$$

$$v = \frac{3k}{C_v \ell}$$

$$@\ 20\text{K} \Rightarrow \quad v = 3 \times 45 \ \frac{\cancel{\text{Watt}}}{\cancel{\text{m}}\,\cancel{\text{K}}} \ \frac{\cancel{\text{K}}\,\text{m}^3}{2.64 \times 10^4 \ \cancel{\text{Joule}}} \ \frac{1}{2.3 \times 10^{-6} \ \cancel{\text{m}}} \simeq 2223.3 \ \text{m/s}$$

$$@\ 273\text{K} \Rightarrow \quad v = 3 \times 6.4 \ \frac{\cancel{\text{Joule/sec}}}{\cancel{\text{m}} \cdot \cancel{\text{K}}} \ \frac{\cancel{\text{K}}\ \text{m}^3}{93 \times 10^4 \ \cancel{\text{Joule}}} \ \frac{1}{67 \times 10^{-10} \ \cancel{\text{m}}} \simeq 3081.3 \ \text{m/s}$$

3.18 not assigned this year, but discussed in lecture-please read

from OMAR, READ CAREFULLY, THE FOLLOWING is TAKEN from ASHCROFT & MERMIN

N.B.: The scattering of x-rays by phonons, treated above from a quantum point of view, may also be viewed as a classical process in which the electromagnetic wave is diffracted from the acoustic wave. The lattice wave, in producing regions of compression and rarefication in the medium, acts as a set of atomic planes from which the x-ray beam suffers Bragg diffraction, the interplanar spacing being equal to the wavelength. From this vantage point, the momentum equation (3.75) is simply the Bragg condition for constructive interference, Eq. (2.47). The energy equation (3.76) follows from the fact that, since the wave is moving, the x-ray beam should suffer a Doppler shift in its frequency. In the case of phonon absorption, the wave is traveling toward the x-ray beam and the shift is positive, while in the process of phonon emission, the wave travels away from the beam and the shift is negative. When the Doppler shift is treated quantitatively, it leads precisely to (3.76), as you may convince yourself.

Consider, then, the interaction of a wave with angular frequency $E/\hbar$ and wave vector $\mathbf{q} = \mathbf{p}/\hbar$, with a particular normal mode of the crystal with angular frequency ω and wave vector $\mathbf{k}$. We suppose that only this particular normal mode is excited; i.e., we consider the interaction of the wave with one phonon at a time. We also ignore for the moment the microscopic structure of the crystal, regarding the normal mode of interest as a wavelike disturbance in a continuous medium. If the disturbance did not move it would present to the incident radiation a periodic variation in density, which would act as a diffraction grating (Figure 24.10), the scattered wave being determined by Bragg's law. However, the disturbance is not stationary, but is moving with the phonon phase velocity which is directed along $\mathbf{k}$ and has magnitude ω/k:

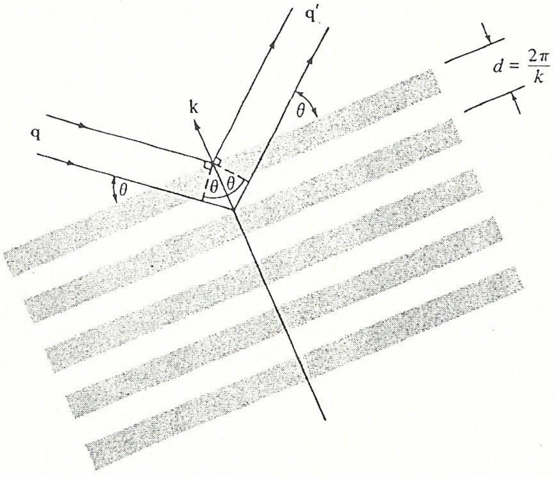


Figure 24.10
Scattering of a neutron by a phonon in a frame of reference in which the phonon phase velocity is zero. The phonon appears as a static diffraction grating; i.e., it results in regions of alternating high and low ionic density. The Bragg condition (p. 97), $m\lambda = 2d \sin\theta$ (m an integer), can be written as

$$\frac{2\pi m}{q} = \frac{4\pi}{k} \sin\theta$$

or

$$mk = 2q \sin\theta$$

or

$$mk = (\mathbf{q}' - \mathbf{q}) \cdot \hat{\mathbf{k}}.$$

Since Bragg reflection is specular (angle of incidence equals angle of reflection) and since the magnitude q' equals the magnitude q, it follows that $\mathbf{q}' - \mathbf{q}$ must be parallel to $\mathbf{k}$, and therefore $\mathbf{q}' - \mathbf{q} = m\mathbf{k}$.

$$\mathbf{v} = \frac{\omega}{k}\hat{\mathbf{k}}. \tag{24.22}$$

This complication can be dealt with by describing the diffraction in the frame of reference that moves with the phase velocity $\mathbf{v}$. In that frame the disturbance will be stationary, and the Bragg condition can be applied. The wave vectors (of both the lattice wave and the incident and scattered waves) are unaltered by a change of frame, since this affects neither the distance between planes of constant phase nor their orientation.[11] However, the frequencies undergo a Doppler shift:

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[11] More precisely, the change in wave vector is a relativistic effect, which we ignore because the phase velocity $\mathbf{v}$ is small compared with c. The Doppler shift formulas (24.23) are also used in their nonrelativistic forms.

$$\begin{aligned} \bar{\omega} &= \omega - \mathbf{k}\cdot\mathbf{v}, \\ \frac{\bar{E}}{\hbar} &= \frac{E}{\hbar} - \mathbf{q}\cdot\mathbf{v}, \\ \frac{\bar{E}'}{\hbar} &= \frac{E'}{\hbar} - \mathbf{q}'\cdot\mathbf{v}. \end{aligned} \qquad (24.23)$$

Since Bragg reflection by a stationary grating leaves the frequency of the incident wave unaltered, $\bar{E}'$ must be equal to $\bar{E}$. The transformation law (24.23) then implies that in the original frame the frequency of the scattered wave must be shifted:

$$\frac{E'}{\hbar} = \frac{E}{\hbar} + (\mathbf{q}' - \mathbf{q})\cdot\mathbf{v}. \qquad (24.24)$$

The change in wave vector under a Bragg reflection has the form:

$$\mathbf{q}' = \mathbf{q} + m\mathbf{k}, \qquad (24.25)$$

where the integer m is the order of the Bragg reflection (as demonstrated in Figure 24.10).[12] This relation holds in either frame, since wave vectors are invariant under change of frame.

Substituting (24.25) into (24.24), we find that the frequency shift in the original frame is given by

$$\frac{E'}{\hbar} = \frac{E}{\hbar} + m\mathbf{k}\cdot\mathbf{v}. \qquad (24.26)$$

If we substitute into (24.26) the explicit form (24.22) for the phase velocity $\mathbf{v}$, we find that

$$E' = E + m\hbar\omega. \qquad (24.27)$$

Equations (24.25) and (24.27) reveal that an mth-order Bragg reflection in the moving frame corresponds to a process that we would describe in the laboratory frame as the absorption or emission of m phonons of a given type. Multiphonon processes involving several normal modes evidently will correspond to successive Bragg reflections from the corresponding moving diffraction gratings.

The wave vector condition (24.25) might appear to lack the arbitrary additive reciprocal lattice vector that is present in the crystal momentum conservation law (24.6). Actually, it is implicit in (24.25) as well, as soon as we acknowledge that the crystal is not a continuum, but a discrete system. Only in a continuum is it possible to assign a unique wave vector **k** to each normal mode. In a discrete lattice the normal mode wave vector is defined only to within an additive reciprocal lattice vector (see page 439).

Thus from the wave point of view the law of energy conservation is simply a statement of the Doppler shift for a wave reflected from a moving diffraction grating; the law of crystal momentum conservation is the Bragg condition for that grating, the additive reciprocal lattice vector expressing the variety of orientations that the grating may be considered to have, due to the discrete periodic nature of the underlying crystal.

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[12] Note that m can have either sign, depending on the side of the grating from which the wave is incident.

[7] $\boxed{3.19}$

$$\Delta\omega = \pm 2n\omega_0 \left(\frac{v_s}{c}\right)\sin\theta$$

<u>Recall</u>

$$\Delta\nu = \frac{\Delta\omega}{2\pi}$$

$$\omega_0 = \frac{2\pi c}{\lambda_0}$$

$$\left.\begin{aligned} \theta &= \pi/2 \\ n &= 1.33 \\ \lambda_0 &= 6328\times10^{-10}\,\text{m} \\ \Delta\nu &= 4.3\times10^{9}\,\text{s}^{-1} \end{aligned}\right\}\ \text{Given}$$

$$\Rightarrow \quad v_s = \frac{\lambda_0\,\Delta\nu}{2n\sin\theta} = \frac{(6328\times10^{-10}\,\text{m})(4.3\times10^{9}\,\text{s}^{-1})}{2(1.33)\sin(\pi/2)}$$

$$= \underline{1023\ \text{m/s}}$$

[8] Redundant with [4]

[9]

$$M \frac{d^2 u_n}{dt^2} = -\alpha \left(2u_n - u_{n+1} - u_{n-1}\right)$$

$$u_n = Ae^{i(qna - \omega t)}$$

$$u_{n+1} = Ae^{i[q(n+1)a - \omega t]} = Ae^{i(nqa - \omega t)} e^{iqa} = u_n e^{iqa}$$

$$u_{n-1} = u_n e^{-iqa}$$

$$M \frac{d^2 u_n}{dt^2} = -\alpha \left(2u_n - u_n e^{iqa} - u_n e^{-iqa}\right)$$

$$M \frac{d^2 u_n}{dt^2} = -\alpha u_n \left[2 - \left(e^{iqa} + e^{-iqa}\right)\right]$$

$$\boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}}$$

$$M \frac{d^2 u_n}{dt^2} = -2\alpha u_n \left(1 - \cos qa\right)$$

$$\boxed{\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}$$

$$M \frac{d^2 u_n}{dt^2} = -4\alpha \sin^2 \frac{qa}{2} \cdot u_n$$